The Quantization of Anomalous Gauge Field Theory

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ABSTRACT

We discuss the so called gauge invariant quantization of anomalous gauge field theory, originally due to Faddeev and Shatashvili. It is pointed out that the further non invariance of relevant path integral measures poses a problem when one tries to translate it to BRST formalism. The method by which we propose to get around of this problem introduces certain arbitrariness in the model. We speculate on the possibility of using such an arbitrariness to build series of non equivalent models of two dimensional induced gravity.

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1. Introduction

The consistent quantization of (classical) gauge invariant field theory requires the complete cancellation of anomalies [1] [2]. Here, "consistent" means that we want not only to require renormalizability (perturbative finiteness), but also unitarity of S-matrix, non-violation of Lorentz invariance etc. Moreover, in physical $4d$ world, anomaly cancellation condition itself often leads to the physical predictions. The well known example is the equality of numbers of quarks and leptons in the Standard Model of Weinberg and Salam.

On lower dimensional (eg. $d = 2$) field theory, the cancellation of anomalies is still the crucial ingredient for the model building. The critical string dimension $d = 26$ is often quoted [3] as the consequence of anomaly free condition for bosonic string (although in this example the cancellation of anomaly does not guarantee full consistency of the model in above sense, due to the presence of tachyons).

In the case of lower dimensional field theory $(d < 4)$, one often tries to quantize a gauge field theory when there is no way of cancelling its anomaly. The classical example of this situation is the attempt to the quantization of chiral Schwinger model by Jackiw and Rajaraman [4] [5]. They have shown that the model can be consistently quantized (=free field theory) even when the gauge invariance is broken through anomaly.

In general there seem to be two ways for attempting the quantization of anomalous gauge field theory:

1) *Gauge non invariant method*

One ignores the breaking of gauge symmetry and try to show that the theory can be quantized even without the gauge invariance. The example of this approach is the above Jackiw-Rajaraman quantization of the chiral Schwinger model. The problem here is that it is not easy to develop the general technics covering wide

class of physically relevant models with anomaly.

2) *Gauge invariant method*

In this case, one first tries to recover gauge invariance by introducing new degrees of freedom. The theory is anomalous when one can not find local counter term to cancel the gauge non invariance due to the one loop "matter" integrals in presence of gauge fields, by making use exclusively of the degrees of freedom (fields) already present in the classical action.

In ref. [6], Faddeev and Shatashvili (FS) have tried to justify the introduction of new degrees of freedom which are necessary to construct the anomaly cancelling counter term. Their argument is based on the idea of projective representation of gauge group. They observe that the appearance of anomaly does not mean the simple breakdown of (classical) gauge symmetry, but it rather signals that the symmetry is realised projectively (this is related to the appearance of anomalous commutators of relevant currents). Such a realization, through projective representations, necessitates the enlargement of physical Hilbert space. Thus they argued that the introduction of new fields in the model is not an ad hoc (and largely arbitrary) construction.

Independently of their "philosophy", the FS method gives the gauge invariant action at the price of introducing the extra degrees of freedom (generally physical). The serious problem of this method is, however, that the gauge invariance thus "forced" upon the theory, does not automatically guarantee the consistency of the theory. This is in contrast with our experience with some $4d$ models such as the Standard Model.

For example, one may apply the FS method to the celebrated case of chiral Schwinger model [4] [5] [5A]. In this case, we have the classical action

$$S_0 = \int \frac{dz \wedge d\bar{z}}{2i} \left[\bar{\psi}_R \gamma_{\bar{z}} (\bar{\partial} + R)\psi_R + \bar{\psi}_L \gamma_z \partial \psi_L + \frac{1}{4} Tr \ F^2 \right]$$

where

$$\psi_{R/L} = \frac{1 \pm \gamma_5}{2}\psi$$

$$R/L = A_1 \pm iA_2, \ F = \bar{\partial}L - \partial R + [R, L]$$

(we are using the euclidian notation).

So is invariant under the gauge transformation

$$\psi_R \rightarrow \psi_R^g = S(g)\psi_R$$

$$\psi_L \rightarrow \psi_L$$

$$A_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}$$

for any $g(z, \bar{z}) \in G$.

The theory is anomalous because the one loop integral

$$e^{-W_R(R)} = \int \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_R exp - \int \bar{\psi}_R \gamma_{\bar{z}}(\bar{\partial} + R)\psi_R$$

is not gauge invariant under

$$R \rightarrow gRg^{-1} + g\bar{\partial}g^{-1}$$

(for any choice of the regularization).

Following FS'technic (see next section) however, one can introduce the local counter term $\Lambda(R, L; g)$, $(g(z, \bar{z}) \in G)$ so that the gauge variation of Λ cancels the non invariance of $W_R(R)$.

There is certain arbitrariness in the choice of Λ but the convenient one is

$$\Lambda(R, L; g) = -\left(\alpha_L(L, g) + \frac{1}{4\pi}\int Tr(RL)\right)$$

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where

$$\alpha_L(L,g) = \frac{1}{4\pi}\Big[-\int \frac{dz \wedge d\bar{z}}{2i} Tr(g^{-1}\bar{\partial}g, L) + \frac{1}{2}\int \frac{dz \wedge d\bar{z}}{2i} Tr(g\partial g^{-1}, g\bar{\partial}g^{-1})$$

$$-\frac{1}{2}\int_0^1 dt \int \frac{dz \wedge d\bar{z}}{2i} Tr(g'\partial_t g'^{-1}, [g'\partial g'^{-1}, g'\bar{\partial}g'^{-1}]) \Big]$$

$$g'(0,z,\bar{z}) = 1, \quad g'(1,z,\bar{z}) = g(z,\bar{z})$$

is the Wess-Zumino-Novikov-Witten action corresponding to the anomaly of left fermion ψ_L, $\bar{\psi}_L$ (α_L is not globally a local action but it is so for "small" $g \simeq 1 + i\xi$). That is, one can write

$$\alpha_L(L,g) = W_L(L^g) - W_L(L)$$

where

$$e^{-W_L(L)} = \int \mathcal{D}\psi'_L \mathcal{D}\bar{\psi}'_L \, exp - \int \bar{\psi}'_L \gamma_z (\partial + L)\psi'_L$$

(note that ψ'_L, $\bar{\psi}'_L$ have nothing to do with ψ_L, $\bar{\psi}_L$ in S_0).

With this choice of counter term, one can show that the theory is equivalent to a) free decoupled fermion ψ_L, $\bar{\psi}_L$ and b) the vector Schwinger model. In fact, the added bosonic degree of freedom $g(z,\bar{z}) \in G$ can be "fermionized" to act as missing ψ'_L, $\bar{\psi}'_L$ with the right coupling to the left component L of gauge field.

However, there is still a point missing in this story. In fact, after introducing the new degree of freedom g, there is no reason to exclude the other type of invariant local counter term such as

$$\frac{a}{4\pi}\int Tr(L^g R^g) = \frac{a}{4\pi}\int Tr\Big((gLg^{-1} + g\partial g^{-1}), (gRg^{-1} + g\bar{\partial}g^{-1})\Big)$$

(one can also attribute it to indefinite - regularization dependent - part of fermionic integral, i.e. $W_R(R) + W_L(L) + \frac{a}{4\pi}\int Tr(RL)$).

It is well known [4] that the arbitrary constant a enters the physical spectrum. For abelian case, $G = U(1)$, the mass square of massive boson is given by

$$m^2 = \frac{e^2 a^2}{a - 1}$$

thus, for $a < 1$, the theory is not consistent although the requirement of gauge invariance is satisfied.

In the fermionized version of the theory [5A], a enters the charges of left and right fermions as

$$e_{R/L} = \frac{e}{2}\left(\sqrt{a - 1} \pm \frac{1}{\sqrt{a - 1}}\right)$$

This means that the condition $a > 1$ is necessary also for the real coupling constant, or the hermitian hamiltonian.

In general, the consistence of the theory can be proved if one can set up the BRST scheme with certain physical conditions at the start, such as hermiticity of the hamiltonian [11].

In what follows, we discuss the possibility of recasting the FS method into BRST formalism, thus facilitating the analysis of the consistency of the theory.

2. Faddeev-Shatashvili method

a) Path integral formalism

We shall briefly describe the Faddeev-Shatshvili (FS) method of quantizing anomalous gauge field theory in the path integral formalism, following the work of Harada and Tsutsui [7], Babelon, Shaposnik and Vialet [8].

Let us take a generic gauge field theory described by the classical action

$$S_0(A, X) = S_G(A) + S_M(X; A) \tag{1}$$

where $\{A(x)\}$ and $\{X(x)\}$ represent respectively gauge fields and "matter fields", gauge invariantly coupled to the former.

The total action S_0 as well as the pure gauge part S_G and the matter part S_M are invariant under the local gauge transformation

$$A \to A' = A^g$$
$$X \to X' = X^g \qquad g(x) \in G. \tag{2}$$

Being anomalous generally means that the one loop matter integral (assumed that $S_M(X, A)$ is quadratic in X)

$$\int \mathcal{D}X e^{-S_M(X,A)} \equiv e^{-W(A)} \tag{3}$$

cannot be regularized in such a way as to preserve the gauge invariance of the functional $W(A)$,

$$W(A^g) - W(A) = \alpha(A; g) \neq 0 \tag{4}$$

Naturally, $\alpha(A; g)$ depends on the regularization used, but there is no way of cancelling it completely by adding some local counter term $\Lambda(A, X)$ to the action.

One can understand eq. (4) as the non invariance of the path integral measure, $\mathcal{D}X$:

$$\mathcal{D}X^g \neq \mathcal{D}X \tag{5}$$

In fact, as shown by Fujikawa [9], one can write the "anomaly equation"

$$W(A^g) - W(A) = \alpha(A; g)$$
$$det\left(\frac{\mathcal{D}X^g}{\mathcal{D}X}\right) = e^{-\alpha(A;g)} = e^{\alpha(A;g^{-1})} \tag{6}$$

In this situation, clearly one can not hope that the usual Faddeev-Popov (FP) ansatz to quantize the theory may go through.

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If one inserts the δ function identity

$$1 = \Delta(A) \int \mathcal{D}g\delta(F(A^g)) \tag{7}$$

where $F(A)$ is a gauge fixing function, into the path integral expression for the partition function

$$Z = \int \mathcal{D}A \int \mathcal{D}X e^{-(S_G(A)+S_M(X;A))}$$

then one obtains

$$
\begin{aligned}
Z &= \int \mathcal{D}A \int \mathcal{D}X \Delta(A) e^{-[S_G(A)+S_M(X;A)]} \int \mathcal{D}g\delta(F(A^g)) \\
&= \int \mathcal{D}A \int \mathcal{D}g \Delta(A) e^{-S_G(A^g)} \int \mathcal{D}X e^{-S_M(X^g;A^g)} \delta(F(A^g))
\end{aligned} \tag{8}
$$

the second equality follows from the gauge invariance of the classical action: $S_0(A^g; X^g) = S_0(A; X)$.

In the case of usual gauge field theory, such as chiral Schwinger model, we can make a series of assumptions on the remaining functional measures $\mathcal{D}A$ and $\mathcal{D}g$.

First, we assume

$$(1) \qquad \mathcal{D}A = \mathcal{D}A^g \tag{9}$$

then with the change of variable $A^g \to A$ and $X^g \to X$ in (7), we get

$$Z = \int \mathcal{D}g \int \mathcal{D}A \Delta\delta(F(A))(A^{g^{-1}}) e^{-S_G(A)} \int \mathcal{D}X e^{-[S_M(X;A)+\alpha(A;g^{-1})]} \tag{10}$$

where we have used eq. (5), i.e. $\mathcal{D}X = \mathcal{D}X^{gg^{-1}} = \mathcal{D}X^g e^{-\alpha(A;g^{-1})}$.

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Further, one can assume for the usual gauge group, the invariance of Haar measure $\mathcal{D}g$, i.e. for any h in G

$$(2) \qquad \mathcal{D}(gh) = \mathcal{D}(hg) = \mathcal{D}g \tag{11}$$

which results, as is well known, in the invariance of the FP factor $\Delta(A)$

$$\Delta(A^{g^{-1}}) = \Delta(A) \tag{12}$$

Thus, we get the expression for Z proposed in ref.s [6] and [7]

$$Z = \int \mathcal{D}g \int \mathcal{D}A \Delta(A) \delta(F(A)) \int \mathcal{D}X e^{-S_{eff}(X,A;g)} \tag{13}$$

with

$$S_{eff}(X, A; g) = S_0(X, A; g) + \alpha(A; g^{-1}) \tag{14}$$

As one can see from eq. (4) the effect of the counter term $\alpha(A; g^{-1})$ is to transform the one loop path integral $W(A)$, eq. (3), to $W(A^{g^{-1}})$, which is trivially gauge invariant under the extended gauge transformation

$$A \to A^h, \quad X \to X^h$$
$$g \to hg \tag{15}$$

and thus the model is invariant up to one loop level.

We have repeated here the above well known manipulations [7] to emphasize the relevance of the invariance conditions 1) and 2) (eq.s (9) and (11)).

In many familiar example, such as the chiral Schwinger model, these conditions are trivially satisfied.

One well known case where these conditions become problematic is the $2d$ induced gravity or off-critical string. In this case, if one fixes the path integral measures $\mathcal{D}\phi$ for the Weyl factor of metric and $\mathcal{D}\sigma$ for the Weyl group element by the invariance under the diffeomorphisms of $2d$ manifold, then they are not invariant under the translations, eg. $\sigma \rightarrow \sigma + \alpha$ (i.e. Weyl transformation). Thus, the path integral measure (i.e. $\mathcal{D}A\mathcal{D}g$) can never be invariant under the whole gauge group

$$G = Diffeo \otimes Weyl$$

b) BRST [10] quantization

A more rigorous strategy to have a consistent formulation of a gauge field theory is to recast it in the BRST formalism. In this way, one may discuss the physically important questions such as the unitariety of S-matrix [11].

In the simpler example like the chiral gauge field theory where the invariance of the measure $\mathcal{D}g\mathcal{D}A$, eq.s (9) and (11), under the gauge transformations are respected, there is no difficulty in setting up the BRST procedure once the anomaly has been removed.

One replaces the "heuristic" FP factor

$$\Delta(A)\delta(F(A)) = det\left(\left.\frac{\delta F(A^h)}{\delta h}\right|_{h=1}\right)\delta(F(A))$$

with BRST gauge fixing term

$$exp - \int \hat{s}(\bar{c}F(A)) = exp - \int \left[BF(A) - \bar{c}\left.\frac{\delta F(A^h)}{\delta h}\right|_{h=1}c\right]$$

where $c, \bar{c}$ are the BRST ghosts corresponding to the gauge group G while B ("Lagrange multiplier") is the Nakanishi-Lautrup field. Under the BRST operator

$\hat{s}$, one has, in particular

$$\hat{s}\bar{c} = B$$

$$\hat{s}B = 0$$

$$(\hat{s}^2 = 0)$$

The counter term $\alpha(A; g)$ cancelling the one-loop anomaly, one can show easily the validity of the Slavnov-Taylor identity

$$\frac{\delta\tilde{\Gamma}}{\delta A}\frac{\delta\tilde{\Gamma}}{\delta K} + \frac{\delta\tilde{\Gamma}}{\delta\Phi_i}\frac{\delta\tilde{\Gamma}}{\delta K_i} + \frac{\delta\tilde{\Gamma}}{\delta c}\frac{\delta\tilde{\Gamma}}{\delta L}$$

$$(\tilde{\Gamma} * \tilde{\Gamma} = 0)$$

(16)

up to one loop.

$\tilde{\Gamma}$ is the generating functional of the one particle irreducible part Γ (with added external source for composite operators) minus the "gauge fixing term" (in (16), A and c are the classical counter parts of the gauge fields A and ghost c, while $\{\Phi_i\}$ are the classical fields for the matter X and newly introduced field g; K, K_i and L are the usual external sources for the gauge variations $\hat{\delta}A, \hat{\delta}\Phi_i$ and δc respectively). One then hopes that it is possible to chose the higher order local counter term in such a way that eq. (16) is satisfied to all orders.

Let us now imagine, however, that the invariance conditions 1) and 2) for the measure $\mathcal{D}A\mathcal{D}g$ (eq.s (9) and (11)) are not satisfied. This means that one should take account of one or both of the following situations:

(1') the condition (1) is not satisfied, i.e. $\mathcal{D}A \neq \mathcal{D}A^g = \mathcal{D}Ae^{-\alpha'(A;g)}$, where $\alpha'(A; g)$ is the "Fujikawa determinant" associated with the non gauge invariance of measure over gauge field itself.

(2') the condition (2) is not satisfied, i.e. $\Delta(A^g) \neq \Delta(A)$.

First of all, the non invariance property 2') means that the factor $\Delta(A)\delta(F(A))$ in eq. (11) must be replaced by $\Delta(A^{g^{-1}})\delta(F(A))$.

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Thus, instead of a BRST gauge fixing term (14) one ends up with

$$\int \hat{s}(\bar{c}F(A) + ln\left(\frac{\Delta(A^{g^{-1}})}{\Delta(A)}\right) \tag{17}$$

The trouble is that one can not transform $-ln\Delta(A)$ into a BRST invariant local term in the action. In fact, the BRST gauge fixed action would appear something like

$$S_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + ln\left(\frac{\Delta(A^{g^{-1}})}{\Delta(A)}\right) + \int \hat{s}(\bar{c}F(A)) \tag{18}$$

The extra one loop term $\alpha'(A; g)$ does not cause any trouble for the BRST scheme to work at least in the example we are interested. One way to push through the BRST scheme may be to replace eq. (18) with

$$S'_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \int \hat{s}(\bar{c}F(A)) \tag{19}$$

It is likely that the effective action (19) leads to a consistent BRST quantization. One may only add that it does not correspond to the path integral method of ref.s [7] and [8] when $\Delta(A^g) \neq \Delta(A)$.

To reconciliate the "path integral" formulation of FS method with BRST scheme, we propose another possibility.

It must be realized that once the new degree of freedom g is admitted in the theory then there is no reason to exclude new local counter terms of the right dimension which are BRST invariant and which may also depend on g. Naturally this will change the model and its "physics", but nevertheless it can remain consistent, in so far as the BRST invariance is maintained.

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Let us then introduce the following counter term in our theory

$$\tilde{\Lambda}_G(A, g; c, \bar{c}, c', \bar{c}', B) = \left[BG(A^{g^{-1}}) - \bar{c}' \frac{\delta G(A^{g^{-1}h})}{\delta h} \bigg|_{h=1} c' \right] - \left[BG(A) - \bar{c} \frac{\delta G(A^h)}{\delta h} \bigg|_{h=1} c \right] \quad (20)$$

where the second pair of "ghosts" $c', \bar{c}'$ are defined as the BRST singlet

$$\hat{\delta}\bar{c}' = 0$$
$$\hat{s}c' = 0 \quad (21)$$

and $G(A)$ is the "pseudo gauge fixing" which is generally different from $F(A)$.

The first term in $\tilde{\Lambda}_G$ is trivially BRST invariant since all the fields involved are either gauge invariant by themselves or appear as invariant combinations. The second term, on the other hand, can be written as

$$\hat{s}(\bar{c}G(A))$$

so it is invariant too.

The effective action now takes the form

$$S_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \int \tilde{\Lambda}_G(A, g; c, \bar{c}, c', \bar{c}', B) + \int \hat{s}(\bar{c}F(A) \quad (22)$$

Note that the gauge freedom of the BRST invariant theory (22) is represented by the (arbitrary) gauge fixing function $F(A)$ while each different choice of "pseudo gauge function" $G(A)$ defines a new model.

Each choice of $G(A)$ then results in a gauge invariant model which must then be gauge fixed by choosing a particular form for $F(A)$. In the limit of singular gauge

$$F(A) \to G(A) \quad (23)$$

the effective action (22) gives the series of models depending on $G(A)$ alone. The

corresponding effective action can be formally written

$$S_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \left[BG(A^{-1}) - \bar{c}' \frac{\delta G(A^{g^{-1}h})}{\delta h}\bigg|_{h=1} c' \right] \quad (24)$$

Note that in (24) the gauge is already fixed (with a singular gauge). To see the gauge invariance property of the model (24), one must go back to eq. (22) with (20), i.e.

$$S_{eff}^{inv} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \int \left[BG(A^{-1}) - \bar{c}' \frac{\delta G(A^{g^{-1}h})}{\delta h}\bigg|_{h=1} c' \right]$$
$$- \int \left[B[F(A) - G(A)] - \bar{c} \frac{\delta}{\delta h}[F(A^h) - G(A^h)]\bigg|_{h=1} c \right] \quad (25)$$

We have seen in this way that the FS method of formulating an anomalous theory within the path integral formalism apparently generates a series of physically distinct and BRST invariant gauge fields theories.

We will discuss the possible candidate for such a scenario in the next section.

3. Two dimensional induced gravity

In this section we would like to apply the FS method of §1 to analyze the quantization problem of $2d$ gravity [15] (off critical string) in conformal gauge [16]. The theory at classical level is defined in term of the Polyakov action

$$S_0 = \sum_{\mu=1}^{d} \int d^2x \sqrt{g} g^{ab} \partial_a X_\mu \partial_b X^\mu \quad (26)$$

where $\{X^\mu(x)\}_{\mu=1,d}$ are the bosonic matter fields coupled to the $2d$ metric g_{ab} (in the string language, the string is immersed in a d-dimensional target space).

We use euclidian metric and introduce the complex coordinates

$$z = x_1 + ix_2$$
$$\bar{z} = x_1 - ix_2$$

The invariant line element can be written as

$$ds^2 = g_{ab}dx^a dx^b = e^\phi |dz + \mu d\bar{z}|^2 \tag{27}$$

Thus, one can conveniently parametrize the metric as

$$g_{zz} = \bar{\mu}e^\phi, \qquad g_{\bar{z}\bar{z}} = \mu e^\phi$$
$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1 + \bar{\mu}\mu}{2}e^\phi$$

In term of the parameters $\mu, \bar{\mu}$ and ϕ the classical action (26) takes the form [15]

$$S_0 = \sum_{\mu=1}^{d} \int \frac{dz \wedge d\bar{z}}{2i} \frac{(\bar{\partial} - \mu\partial)X_\mu(\partial - \bar{\mu}\bar{\partial})X^\mu}{1 - \mu\bar{\mu}}$$

It is understood that μ and $\bar{\mu}$ are constrained by

$$|\mu|^2 < 1$$

The classical action S_0 is invariant under the gauge group G which is the semidirect product of Diffeomorphisms (general coordinate transformations) and Weyl transformations. These symmetry groups imply respectively:

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1) the symmetry under the general coordinate transformation

$$z \to z' = f(z, \bar{z})$$
$$\bar{z} \to \bar{z}' = \bar{f}(z, \bar{z})$$

(28)

where the relevant fields transform as follows

$$X^{\mu}(z, \bar{z}) \to X^{\mu'}(z', \bar{z}') = X^{\mu}(z, \bar{z}) \quad (scalar)$$
$$\mu(z, \bar{z}) \to \mu'(\bar{z}', \bar{z}') = -\frac{\bar{\partial}f - \mu \partial f}{\partial \bar{f} - \mu \partial \bar{f}}(z, \bar{z})$$
$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \phi(z, \bar{z}) + ln\frac{(\bar{\partial}\bar{f} - \mu \partial \bar{f})(\partial f - \bar{\mu}\bar{\partial}f)}{D_f^2}$$

(29)

where

$$D_f = det \begin{pmatrix} \partial f & \partial \bar{f} \\ \bar{\partial}f & \bar{\partial}\bar{f} \end{pmatrix}$$

2) The symmetry under the local rescaling of the $2d$ metric

$$g_{ab} \to e^{\sigma} g_{ab}$$

or in term of the $\mu, \bar{\mu}$ and ϕ variables

$$\mu \to \mu, \quad \bar{\mu} \to \bar{\mu}, \quad \phi \to \phi + \sigma$$

(30)

It is well known that the theory is anomalous, i.e. one can not regularize the path integral in a way that conserves the whole $G = Diffeo \times Weyl$ group.

One can see this easily, examining the matter integral measure $\mathcal{D}X^{\mu}$. With the simplest (translationally invariant or "flat") regularization $\mathcal{D}_0 X^{\mu}$, one has

$$\prod_{\mu=1}^{d} \int \mathcal{D}_0 X^{\mu} e^{S_0(X, \mu, \bar{\mu})} = exp - \frac{d}{24\pi}[W(\mu) + \bar{W}(\bar{\mu})]$$

(31)

where $W(\mu)$ is the Polyakov's "light cone gauge" action [13].

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This is naturally Weyl invariant (S_0 does not contain the variable ϕ). On the other hand, it is equally clear that one has lost diffeomorphism's invariance, since the invariance under general coordinate transformations means

$$\delta W(\mu) = 0 \tag{32}$$

under $\delta\mu = (\bar{\partial} - \mu\partial + \partial\mu)(\epsilon + \mu\bar{\epsilon})$, which corresponds to the infinitesimal version of eq.s (29) with $f(z, \bar{z}) = \epsilon(z, \bar{z}), \bar{f}(z, \bar{z}) = \bar{\epsilon}(z, \bar{z})$.

Eq. (32) is equivalent to the functional differential equation

$$(\bar{\partial} - \mu\partial - 2\partial\mu)\frac{\delta W}{\delta\mu(z, \bar{z})} = 0$$

A well known computation [16] gives, instead,

$$(\bar{\partial} - \mu\partial - 2\partial\mu)\frac{\delta W}{\delta\mu(z, \bar{z})} = \partial^3\mu \neq 0 \tag{33}$$

Thus, $\mathcal{D}_0 X^\mu$ can not be invariant under diffeomorphisms. One can define the diffeomorphisms invariant measure $\mathcal{D}_{Diffeo} X^\mu$ by introducing the local counter term

$$\Lambda(\mu, \bar{\mu}, \phi) = -\frac{1}{2}\int \frac{dz \wedge d\bar{z}}{2\pi}\Big[\frac{1}{1 - \mu\bar{\mu}}[(\partial - \bar{\mu}\bar{\partial})\phi(\bar{\partial} - \mu\partial)\phi \\ - 2(\bar{\partial}\bar{\mu}(\bar{\partial} - \mu\partial) + \partial\mu(\partial - \bar{\mu}\bar{\partial}))\phi] + F(\mu, \bar{\mu})\Big] \tag{34}$$

where $F(\mu, \bar{\mu})$ is a local function of μ and $\bar{\mu}$ only. We do not need the explicit form of F [17].

The new effective action

$$W_{cov}(\mu, \bar{\mu}, \phi) = W(\mu) + \bar{W}(\bar{\mu}) + \Lambda(\mu, \bar{\mu}, \phi)$$

is invariant under diffeomorphisms.

One can write $W_{cov}(\mu, \bar{\mu}, \phi)$ compactly in the form

$$W_{cov}(\mu, \bar{\mu}, \phi) = \int \frac{dz \wedge d\bar{z}}{2\pi} \frac{(\partial - \bar{\mu}\bar{\partial})\Phi(\bar{\partial} - \mu\partial)\Phi}{1 - \mu\bar{\mu}} = \int d^2x \sqrt{g} g^{ab} \partial_a \Phi \partial_b \Phi \qquad (35)$$

where $\Phi = \phi - ln\partial\zeta\bar{\partial}\bar{\zeta}$ and $\mu = \frac{\bar{\partial}\zeta}{\partial\zeta}$ (Beltrami differentials). Non local (with respect to μ and $\bar{\mu}$) parameter $\zeta(z, \bar{z})$ is Polyakov meson field (13) in $2d$ gravity.

One characterizes the diffeomorphisms invariant measure $\mathcal{D}_{diffeo}X^\mu$ by

$$\prod_{\mu=1}^{d} \int \mathcal{D}_{diffeo}X^\mu e^{S_0(X,\mu,\bar{\mu})} = exp - \frac{d}{24\pi}W_{cov}(\mu, \bar{\mu}, \phi) \qquad (36)$$

(One can understand the appearance of ϕ field, which is absent in the classical action, as due to the introduction of a covariant regularization: Λ_{cov}, $ds^2 \sim e^\phi |dz|^2 > \Lambda_{cov}^2$).

Following for instance DDK [14], in what follows we consistently make use of the diffeomorphisms invariant measure. Thus, except when indicated explicitly otherwise,

$$\mathcal{D}X^\mu \equiv \mathcal{D}_{Diffeo}X^\mu \qquad (37)$$

and more generally $\mathcal{D}\varphi \equiv \mathcal{D}_{Diffeo}\varphi$ for any other filed φ.

Evidently, the diffeomorphisms invariant measure $\mathcal{D}X^\mu$ can not be invariant under the Weyl transformation

$$\phi \rightarrow \phi + \sigma$$

Thus, one establishes that the theory is G anomalous.

(Faddeev-Shatashvili method)

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Having seen that our model for $2d$ gravity is anomalous, one would like to apply to it the FS method of "gauge invariant" quantization of §1. As in §1, we "preestablish" the gauge choice for the full group $G = Diffeo \times Weyl$

$$
\begin{aligned}
\mu &= \mu_0 \\
\bar{\mu} &= \bar{\mu}_0 \quad diffeomorphisms \\
F(\phi) &= 0 \quad Weyl
\end{aligned}
\tag{38}
$$

Since our regularization preserves the diffeomorphisms we assume that the gauge fixing problem (with relevant "b, c" ghosts) for diffeomorphisms has been already taken care for.

To deal with anomalous Weyl symmetry, we have to introduce an extra degree of freedom, a scalar field $\sigma(z, \bar{z})$, corresponding to the element of Weyl symmetry group $g = e^{\sigma(z,\bar{z})}$.

The anomaly cancelling counter term suggested by FS is then given by

$$
\begin{aligned}
\alpha(\mu, \bar{\mu}, \phi; -\sigma) &= W_{cov}((\mu, \bar{\mu}, \phi - \sigma) - W_{cov}(\mu, \bar{\mu}, \phi) = \\
&= -\frac{1}{2} \int \frac{dz \wedge d\bar{z}}{2i} \frac{1}{1 - \mu\bar{\mu}} [(\partial - \bar{\mu}\bar{\partial})\sigma(\bar{\partial} - \mu\partial)\sigma + 2(\partial - \bar{\mu}\bar{\partial})\sigma(\bar{\partial} - \mu\partial)\phi \\
&\quad - 2(\bar{\partial}\bar{\mu}(\bar{\partial} - \mu\partial) + \partial\mu(\partial - \bar{\mu}\bar{\partial}))\phi]
\end{aligned}
\tag{39}
$$

Note that the non local part of W_{cov} is cancelled and $\alpha(\mu, \bar{\mu}, \phi; -\sigma)$ is perfectly local. Naturally, one needs the counter term α for each covariant one loop integral corresponding not only to the matter field $\{X^\mu\}_{\mu=1}^d$, but also to the diffeomorphism ghosts, b, c and $\bar{b}, \bar{c}$, as well as to the ϕ field contained in $W_{cov}(\mu, \bar{\mu}, \phi)$.

Thus, the effective action in sense of §2 is given by

$$
S_{eff} = S_0(X, \mu, \bar{\mu}) + S_{gf}^{(d)}(b, c, \bar{b}, \bar{c}, B, \bar{B}, \mu, \bar{\mu}) + \gamma' \alpha(\mu, \bar{\mu}, \phi; -\sigma)
\tag{40}
$$

where $S_{gf}^{(d)}$ is the gauge fixing term with respect to the non anomalous diffeomorphism symmetry.

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As explained above, the coefficient γ' is contributed by all the relevant fields, that is $\{X^\mu\}_{\mu=1}^d \Rightarrow d, (b, c, \bar{b}, \bar{c}) \Rightarrow -26, \phi \Rightarrow 1$, which gives $\gamma' = \frac{d-26+1}{24\pi} = \frac{d-25}{24\pi}$.

Note that the contribution of ϕ field is due to the fact that $\mathcal{D}_{Diffeo}\phi \neq \mathcal{D}_0\phi$, or in the terminology of §2, that one needs the "second" FS counter term "$\alpha'(\phi; \sigma)$".

One can now write down the partition function Z with the FS prescription (within the path integral formalism of ref. [7], see eq. (8) of §1). Integrating out the "matter fields" $(X^\mu, b, c, \bar{b}, \bar{c},)$, one has

$$Z \sim \int \mathcal{D}\sigma \mathcal{D}\phi \Big[exp - \gamma' \int \frac{dz \wedge d\bar{z}}{2i} \frac{1}{1 - \mu_0\bar{\mu}_0} ((\partial - \bar{\mu}_0\bar{\partial})(\phi - \sigma)(\bar{\partial} - \mu_0\partial)(\phi - \sigma)$$
$$- 2(\bar{\partial}\bar{\mu}_0(\bar{\partial} - \mu_0\partial) + \partial\mu_0(\partial - \bar{\mu}_0\bar{\partial}))(\phi - \sigma)) \Big] \Delta(\phi - \sigma)\delta(F(\phi))$$

$$(41)$$

where the local action in the exponential is essentially a Liouville action $S'_L(\phi')$, $(\phi' = \phi - \sigma)$. The last two factors come from the δ function insertion

$$\Delta(\phi) \int \mathcal{D}\sigma \delta(F(\phi + \sigma)) = 1 \tag{42}$$

Note that, since $\mathcal{D}\sigma \equiv \mathcal{D}_{Diffeo}\sigma \neq \mathcal{D}_0\sigma$ ($\mathcal{D}_0\sigma$ "flat" measure)

$$\Delta(\phi - \sigma) \neq \Delta(\phi) \tag{43}$$

Formally, one can write the $\Delta(\phi - \sigma)$ factor as a local action with the help of the "Weyl ghosts" ψ and $\bar{\psi}$

$$\Delta(\phi - \sigma) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} exp - \int \bar{\psi}\frac{\delta F(\phi - \sigma)}{\delta\phi}\psi \tag{44}$$

(BRST procedure)

The path integral argument of §1 is at best heuristic. It may suggest the possible models but one can not prove in this way their consistency.

As it has been argued in §1, one may start a more precise discussion after setting up the BRST quantization procedure. The BRST properties of the type of models we are dealing with here, have been studied in details for the critical case, i.e. for $d = 26$, where the theory is not anomalous. In ref. [15], the BRST transformation properties of the fields are given. They may be used to study our (off critical) model.

One has (see eq. (29))

$$
\begin{aligned}
\hat{\delta} X^\mu &= (\xi \cdot \partial) X^\mu \\
\hat{\delta} \mu &= (\bar{\partial} - \mu \partial + \partial \mu) c \\
\hat{\delta} \phi &= \psi + (\xi \partial) \phi + (\partial \xi) + \mu \partial \bar{\xi} + \bar{\mu} \bar{\partial} \xi \\
\hat{\delta} \xi &= (\xi \cdot \partial) \xi \\
\hat{\delta} c &= c \partial c \\
\hat{\delta} \psi &= (\xi \cdot \partial) \psi
\end{aligned}
\tag{45}
$$

where $\xi \cdot \partial$ means $\xi \partial + \bar{\xi} \bar{\partial}$.

Here $\hat{\delta}$ stands for the both Weyl and diffeomorphism symmetries. The diffeomorphism ghosts $c, \bar{c}$ are related to the original $(\xi, \bar{\xi})$ (corresponding to $\delta z = \epsilon(z, \bar{z}), \delta \bar{z} = \bar{\epsilon}(z, \bar{z})$) by

$$
\begin{aligned}
c &= \xi + \mu \bar{\xi} \\
\bar{c} &= \bar{\xi} + \bar{\mu} \xi
\end{aligned}
\tag{46}
$$

To eq. (45), we must add the transformation of the auxiliary field $\sigma(z, \bar{z})$. Since σ must be a scalar with respect to diffeomorphisms one has

$$
\hat{\delta} \sigma = \psi + (\xi \cdot \partial) \sigma
\tag{47}
$$

Together with the formulae in eq.s (45) to (47), one consistently finds

$$
\hat{\delta}^2 = 0
\tag{48}
$$

One should add also the diffeomorphisms anti ghost $(b, \bar{b})$ and Weyl anti ghost

$\bar{\psi}$ with the corresponding Nakanishi-Lantrup fields B and D. Their transformation properties are

$$\hat{s}b = B, \ \ \hat{s}\bar{b} = \bar{B}, \ \ \hat{s}\bar{\psi} = D$$
$$\hat{s}B = \hat{s}\bar{B} = \hat{s}D = 0 \tag{49}$$

We have seen, however, that the Faddeev-Popov factor $\Delta(\phi)$ is not Weyl invariant (43). Thus, according to the result of §1, one needs to correct the effective action S_{eff} by modifying the factor $\Delta(\phi - \sigma)\delta(F(\phi))$ into a BRST gauge fixing term. As we have seen in §1, such a prescription is not unique. Formally, any action of the form

$$BRST(invariant) + \hat{s}(\psi F(\phi))(BRST exact)$$

will do the job.

Now the factor $\Delta(\phi - \sigma)\delta(F(\phi))$ can be rewritten in the form

$$exp - \int \left(\mathcal{D}F(\phi) + \bar{\psi}'\frac{\delta F}{\delta \phi}(\phi - \sigma)\psi' \right)$$

Thus, in order to follow this expression as close as possible, we suggest to add a counter term of the form of eq. (20) in §1

$$\tilde{\Lambda}_G(\phi, \sigma; \psi, \bar{\psi}, \psi', \bar{\psi}', D) = \left[DG(\phi - \sigma) + \bar{\psi}'\frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \right] - \left[DG(\phi) + \bar{\psi}\frac{\delta G}{\delta \phi}(\phi)\psi \right] \tag{50}$$

where we have introduced the function $G(\phi)$ to distinguish it from the true gauge fixing term $s(\bar{\psi}F(\phi))$. The new fields ψ' and $\bar{\psi}'$ in eq. (50) (c' and $\bar{c}'$ in eq. (20)) are Weyl singlet and transform as

$$\hat{\delta}\bar{\psi}' = 0$$
$$\hat{\delta}\psi' = (\xi \cdot \partial)\psi' \tag{51}$$

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With the addition of the counter term $\tilde{\Lambda}_G$, the effective action now reads

$$\tilde{S}_{eff} = S_L''(\phi - \sigma) + \int \tilde{\Lambda}_G(\phi, \sigma; \psi, \bar{\psi}, \psi', \bar{\psi}', D) + \int \hat{s}(\bar{\psi} F(\phi))$$

$$= S_L''(\phi - \sigma) + \int \left[DG(\phi - \sigma) + \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \right] + \int \hat{s}(\bar{\psi}(F - G)(\phi))$$

$$(52)$$

The expression for $\tilde{S}_{eff}$ contains two arbitrary functions $F(\phi)$ and $G(\phi)$. Their roles are completely different. While $F(\phi))$ is a genuine gauge fixing function, each choice of $G(\phi)$ actually defines a new model.

Naturally, the "series" of models (at arbitrary gauge) includes the familiar cases. For example, if one fix the model by choosing

$$G = 0$$

one reproduces the physically equivalent formulations of DDK model.

Alternatively, for any given G, one may consider the singular gauge limit

$$F \to G$$

In this limit the model formally corresponds to the action

$$\tilde{S}_{eff} = S_L''(\phi - \sigma) + \int \left[DG(\phi - \sigma) + \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \right] \qquad (53)$$

This is the type of model treated in ref. [18]. One may further add the BRST invariant term $-\frac{\lambda}{2} \int D^2$ and transform $\tilde{S}_{eff}$ into

$$\tilde{S}'_{eff} = S_L''(\phi - \sigma) + \int \left[\frac{1}{2\lambda} G^2(\phi - \sigma) + \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \right] \qquad (54)$$

Eq. (53) (or (54)) seems to be the closest BRST quantizable approximation to

the consequence of FS prescription, i.e. the insertion

$$1 = \Delta(\phi) \int \mathcal{D}\sigma \delta(G(\phi + \sigma)) \qquad (55)$$

In ref. [18], and in some later works, the choice

$$G(\phi) = R(\phi) - R_0 \qquad (56)$$

with R the scalar curvature, has been made. Using (56), the effective action (54) becomes

$$\tilde{S}'_{eff}(\phi' = \phi - \sigma), \psi', \bar{\psi},) = S''_L(\phi') + \int \left[\frac{1}{2\lambda}(R(\phi') - R_0)^2(\phi - \sigma) + \bar{\psi}'\frac{\delta R}{\delta \phi}(\phi - \sigma)\psi' \right] \qquad (57)$$

Note that the model defined by (57) is fully interacting. In particular a) the presence of propagating ψ' and $\bar{\psi}'$ fields and b), more importantly, the presence of $\psi', \bar{\psi}'$ and ϕ' (Yukawa) interaction in (57), change the parameters in the Liouville type action $S''_L(\phi')$. Such a change, which affects the low energy dynamics of (57), can not be calculated exactly. It is not easy even to develop a systematic perturbation expansion [20]. We believe [18] [19] that the modification represented by eq. (57) may result in deviations from the classical DDK result, when one uses (57) to calculate such physical quantities as string tension and anomalous dimension.

Lastly, it must be mentioned that the BRST invariant term

$$\int \bar{\psi}'\frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \qquad (58)$$

in (53) could also be obtained from the alternative gauge fixing

$$S_{gf} = \int \hat{s}\left[\bar{\psi}\frac{\delta G}{\delta \phi}(\phi - \sigma)\sigma \right] \qquad (59)$$

In this case, one can dispense with the extra BRST invariant (for Weyl trans-

formation) ψ' and $\bar{\psi}'$ degrees of freedom. The gauge fixing function is

$$F(\phi, \sigma) = \frac{\delta G(\phi - \sigma)}{\delta \phi}(\phi - \sigma)\sigma \tag{60}$$

It looks as if this model is gauge equivalent to the DDK model, since the gauge choice $G(\phi) = \phi$ gives the effective action

$$S_{eff} = S'_L(\phi - \sigma) + \int (\psi\bar{\psi} + D\sigma) \sim S'_L(\phi) \quad (\sigma = 0) \tag{61}$$

The Liouville action S'_L here is identical to eq. (42) without further renormalization (eq. (57) is a free field action).

4. Conclusion

In this note, we have tried to analyze further consequences of the Faddeev-Shatshvili method of quantizing anomalous gauge fields theories.

In contrast with other authors [5A], we did not try to show the equivalence with the "gauge non invariant" method of which the Jackiw-Rajaraman treatment of the chiral Schwinger model is a distinguished example. On the contrary, we have argued that, in certain cases of physical interest, the FS method can be used to generates new models.

The series of "new" $2d$ gravity models proposed here includes the models in ref.s [18] [19] as well as the Kawai-Nakayama type $(R - R_0)^2$ (or R^2) models [21] [22].

To see if the possibility of enlarging in this way the $2d$ (induced) gravity models really throws some light on the famous problem of the $d = 1$ barrier in $2d$ gravity, we need a more thorough analysis of the consistency of these models as well as a

better understanding of their physical consequences.

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